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OCTOBER 1957

THE DYNAMICS OF SHELL

HARRY L. REED, Jr.

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HLReed,Jr/eeb
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THE DYNAMICS OF SHELL

ABSTRACT

The exterior ballistic motion of shell is considered by the methods of classical mechanics. These methods prove to be powerful tools for a qualitative analysis of the non-linear motion of shell and, in certain special cases, for obtaining quantitative results with a minimum of computational difficulty. The solution thus obtained for the aircraft gunfire problem has proven to be especially useful.

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INTRODUCTION

Interest in the general motion of shell with nonlinear aerodynamics and dynamics has been academic until only recently. While the first attempt at a study of the general motion was made by Fowler and Lock[2] in 1922 (two years after their presentation of the first definitive work on the subject of linear motion[1]), their considerations showed nothing worth intensive study since they only considered nonlinearity in the overturning moment. The experimental methods were too weak to disclose any good data on the Magnus effect[3].

I. L. Synge[4] and C. H. Kebby[5] have made similar analyses of the shell as a "top". The latter reference is extremely detailed, giving all possible cases for the "Fowler moment".

Not only was the experimental evidence lacking, but also the practical interest in large yaw motion. For most weapons, to be effective the shell must be kept in the small yaw region. This is a problem for linearized theory. Two notable exceptions to this premise are high angle artillery fire and bomber defense gun fire. The first problem, popularized by the requirements of jungle and hill warfare, would most probably be best solved by avoiding large summital yaws, but this does not appear possible. The second problem has large yaws almost by definition, namely the launching of shell into crosswinds whose velocity is comparable to the muzzle velocity. The second problem with its urgency has supplied the necessary funds and priorities for truly broaching the subject of large yaw.

In 1952, L. H. Thomas[6] considered the motion of shell as a problem in classical perturbation theory. Unfortunately the heyday of classical techniques was at the turn of the century, and few if any present day ballisticians were prepared to cope with the mechanical sophistication of the paper. This is especially unfortunate since the classical methods have proven to be so powerful in handling the orbital problems of the astronomer, and the problems of orbits with their multiple periodicities are not unlike the problems of the rigid body motion of shell.

Fortunately there is nothing about the perturbation methods of classical mechanics which requires a deep understanding of complicated notions of physics. This was stressed by Dr. Thomas in conversation with the author. That is, the almost-constants can be arrived at by either considering the canonical transformations of Hamiltonian theory, the conservation of energy and momentum, or just ad hoc properties of the differential equations. In fact it is the latter approach which C. H. Murphy[7] applied so successfully to the problem at hand.

It is the purpose of this paper to reproduce the results of Thomas by appealing to direct considerations of the equations of motion. At the same time it is desirable to correlate the direct approach with the body of Hamiltonian theory so that the numerous methods of the astronomers may be made more available to the ballisticians. Therefore, the first section is concerned with the general theory of canonical transformation and secular variables. The remainder of the paper develops the theory of the motion of shell independently of the first section but with identifying references to the first section.

A departure from Thomas' paper is the use of the function defined by equation 6.3 as the fundamental motion rather than a sinusoidal approximation. The author prefers to make the trigonometic approximations as late as possible in the averaging process. Sections 12, 13, and 14 are extensions of Thomas' paper.

One particular limitation of this paper is the omission of the effects of gravity. While this deletion is not important in the case of flat fire and rather desirable for the sake of clarity, there is a large area of interest (Howitzer fire) in which the effects of gravity are of prime importance. It is in this area that the author hopes to extend these methods.

1. HAMILTONIAN APPROXIMATION THEORY

The basis for classical perturbation theory, and for that matter any approximation theory, is the study of the variation of what would be constants in a simpler but similar system. We shall consider two

types of systems: a simple rigid body with a conservative moment of force and additional non-conservative aerodynamic forces and moments of force and a compound shell with a conservative moment of force.

For the first type we assume that everything possible has been put into Hamiltonian form. That is

$$\begin{aligned} \dot{p}_i &= -\ \partial H/\partial q_i + f_i \\ \dot{q}_i &= \ \ \partial H/\partial p_i + g_i \end{aligned} \qquad 1.1$$

where the q's are the generalized coordinates, p's are their conjugate moments, H is the Hamiltonian, and the f's and g's represent those forces and moments which defy a Hamiltonian formulation.* H contains the kinetic energy of the shell and those aerodynamic forces which are conservative (usually the overturning moment).

We further assume that the effect of the f's and g's is to produce a small secular change in the motion which is typified in any small region of time by H. This is to say, the conservative motion at any instant gives the frequencies and amplitudes of oscillations in the system while the non-conservative terms produce long term changes in the parameters of the conservative system. To isolate these parameters of the conservative system, we shall introduce new variables such that the coordinates are almost cyclic and hence the conjugate momenta are almost constant. These new momenta will be the desired parameters.

We make a canonical change of variables as follows:

$$\text{Let} \qquad \begin{aligned} P_1 &= H \\ P_i &= p_i \qquad i \neq 1 \end{aligned} \qquad 1.2$$

*For a general discussion of the Hamiltonian formulation see References 8 and 9. These are only two of many possible references.

We shall find that for our purposes the case with all but one variable's being cyclic is of particular interest. That is $H = H(p_1, \ldots, p_n, q_1)$. To complete the transformation we let

$$p_1 = G(H, p_2, \ldots, p_n, q_1)$$

and define a generating function

$$F = \sum_{i=2}^{n} q_i P_i + \int^{q_1} G(P_1, \ldots, P_n, x)dx \qquad 1.3$$

which gives the transformation equations

$$p_i = \partial F/\partial q_i$$

$$Q_i = \partial F/\partial P_i$$

The generating function gives the following transformation for the new coordinates:

$$Q_i = q_i + \int^{q_1} \partial G/\partial P_i \; dx \qquad i \neq 1$$

$$Q_1 = \int^{q_1} \partial G/\partial P_1 \; dx \; . \qquad 1.4$$

Noting that

$$\partial H/\partial Q_i = 0$$

$$\partial H/\partial P_i = \delta_{i1} \; , \quad \text{(the Kronecker delta)},$$

we write the new Hamiltonian equations as

$$\dot{P}_i = (\partial P_i/\partial p_j)f_j + (\partial P_i/\partial q_j)\, g_j$$

$$\dot{Q}_i = \delta_{i1} + (\partial Q_i/\partial p_j)\, f_j + (\partial Q_i/\partial q_j)\, g_j \qquad 1.5$$

where summation is to be taken over repeated indicies.

The new generalized coordinates are the time in the conservative system for i=1 and the perturbations produced in the original coordinates (q) by the introduction of the non-conservative terms. It should be pointed out that no approximations have been made up to this point.

Allowing that the non-conservative terms may be considered as small and hence that the Q's for $i \neq 1$ may be taken as zero and $Q_1 = t$, we obtain the secular equations

$$\dot{P}_i = h_i \, (P_1, \; . \; . \; . \; , \; P_n, t) \tag{1.6}$$

from equations 1.5 by substituting the solution of the conservative system in the right hand side. A further simplification can be effected by replacing the right hand side by its average over a period of motion giving:

$$\dot{P}_i = (1/T) \int_{t - T/2}^{t + T/2} h_i \, (P_1, \; . \; . \; . \; , P_n, \; y) \, dy \; , \tag{1.7}$$

the P's being assumed constant during the averaging process. T is the period of motion.

Should the frequencies of motion be of particular interest, a suitable choice of variables would be the action-angle variables. If on the other hand the magnitude of the motion is of first importance, the roots of the energy equation would be more appropriate. The latter will be the case for most of the further discussion.

The systems of the second type, mentioned at the beginning of this section, will be considered in the last section since, in breaking faith somewhat with the introduction, the development will be done in the Hamiltonian manner.

2. THE EQUATIONS OF MOTION

As was mentioned in the introduction, the following development will be performed in a direct fashion, i.e. by the use of familiar equations of motion and simple algebraic operations thereon. Nevertheless the development will parallel the Hamiltonian method outlined in section 1, and at appropriate points reference will be made to the preceding outline. The object of the direct development is twofold.

First the complicated force system is hard to identify in terms of generalized coordinates; second the direct approach will be easier for the uninitiated to follow, while, at the same time, giving a view into the inner workings of the theory if continual reference is made to section 1 and Textbooks of classical mechanics.

The starting place will be the equations of motion in almost the same form as presented in reference 10. The major difference is in the definitions of the aerodynamic forces and moments for which the axial component of the velocity is used in the reference while the total speed is used herein. Also we shall use the arc length of the trajectory for the independent variable rather than the integral of the axial component of the velocity*. The effects of drag and spin deceleration are not considered although they may be added with little more than an increase in bookkeeping. The effects of gravity are not considered, which limits the discussion to "flat fire".

Without further discussion the equations of motion are taken to be:

$$\lambda' = i\mu \ell - \ell J_L \lambda \qquad 2.1$$

$$\mu' = iA\nu\mu/B - (\nu J_T + iJ_M) k_2^{-2} \lambda - J_H k_2^{-2} \mu$$

where

λ	Complex orientation of the velocity vector with respect to the shell axis
μ	Complex transverse angular velocity of the shell in radians/caliber of travel
$'$	d/ds
s	arc length of the trajectory in calibers
δ	angle of yaw
ℓ	$\cos \delta$
$\sin \delta$	$\lvert\lambda\rvert$
A	axial moment of inertia
B	transverse moment of inertia

*Actually, the same definitions as we use were used in the earlier but probably not so widespread work of Kelley and McShane, On the Motion of a Projectile with Small or Slowly Changing Yaw, BRL Report 446, 1944.

k_1 axial radius of gyration in calibers

k_2 transverse radius of gyration in calibers

ν axial spin in radians/caliber of travel

J $\rho d^3 K/m$

ρ density of air

d diameter of shell

m mass of shell

K_L lift coefficient

K_T Magnus moment coefficient

K_H damping in pitch coefficient

K_M static moment coefficient

3. THE VARIABLES H, ϕ, and ℓ

To reach the starting place of section one, we must change the variables of section two into canonical variables. Although we shall go directly to a set of variables containing H (as in 1.5), it is worth while indicating the intermediate canonical variables. Using the definition of Eulerian angles in reference 8 with $\delta = \theta$, one can obtain a set of canonical variables. The q_1 of section one would be; $q_2 = \phi$; $q_3 = \psi$; $p_1 = p_\delta$; $p_2 = P_2 = p$; $p_3 = P_3 = p_\psi$. In what follows we shall use the dimensionless $\bar{\phi}$ and ν in place of P_2 and P_3.

The normalization factor is $A\nu$. In this section a bar over a symbol represents the complex conjugate.

We define a dimensionless energy (or Hamiltonian)

$$H = \frac{1}{2}\left(\frac{B}{A\nu}\right)^2 \mu\bar{\mu} + V \qquad 3.1$$

which is the sum of the kinetic energy of the transverse angular motion and the potential energy of the overturning (static) moment

$$V = \int \frac{d\ell}{4S} \qquad 3.2$$

where

$$S = \text{stability factor} = (A\nu)^2/(4B^2 J_M k_2^{-2}) \ .$$

The normalization factor is $(A\nu)^2/B$.

We further define a dimensionless angular momentum which is the component of the shell's angular momentum about the velocity vector

$$\not{\emptyset} = \frac{B}{2A\nu} \quad (\lambda\bar{\mu} + \bar{\lambda}\mu) + \ell. \qquad 3.3$$

The variables H and $\not{\emptyset}$ are essentially those used by L. H. Thomas.

Using equations 2.1, the above definitions, and some straightforward algebra, one obtains the following as the equations of motion in the new variables:

$$H' = -J_H k_2^{-2} (2H-2V) - J_T k_1^{-2} (\not{\emptyset} - \ell) + \frac{1}{4S} J_L (1 - \ell^2)$$

$$\not{\emptyset}' = -(\ell J_L + J_H k_2^{-2}) (\not{\emptyset} - \ell) - (J_T k_1^{-2} - J_L) (1 - \ell^2) \qquad 3.4$$

$$\ell' = \pm \frac{A\nu}{B} \sqrt{(1 - \ell^2) (2H-2V) - (\not{\emptyset}-\ell)^2} + J_L (1-\ell^2) .$$

The radical in the right-hand member of the last equation is equal to G of the first section multiplied by $\sin^2 \delta$. Of course, ℓ is not a canonical coordinate, and a new variable Θ, which is more closely related to the coordinate canonical to H, will be introduced in section 6. However ℓ will continually occur in our considerations, and a word about the use of the cosine of the angle of yaw in the definition of the aerodynamic coefficients is in order.

It is common practice to represent the aerodynamic coefficients as even functions of the sine of the angle of yaw. This practice is satisfactory for angles less than 90°. Beyond 90° this representation gives the value of the coefficient for δ to be that of $180^\circ - \delta$ which is generally incorrect. The general expression for an even function (in angle), which is not completely pathological, is a Fourier cosine series which, if the function is analytic, can be represented by a power series involving both even and <u>odd</u> powers of the cosine. Not only does the cosine appear as a logical variable in the dynamics but also in the aerodynamics.

Of additional interest is the ease with which a cosine series can be fitted to experimental data by trignometric interpolation[11] and further the ease in converting a cosine Fourier series to a cosine power series.

The advantage of the variables H and $\bar{\phi}$ is that they are constant for the case in which the motion is that of a top with a generalized overturning moment and no other forces. However, it is just that top motion which makes the equations so difficult to handle. Essentially we now have variables with most of the high frequency top motion stripped out.

4. THE FUNCTION $f(\ell)$

It was noted in the last section that the radical on the right of the last of equations 3.4 was the function G of section one multiplied by the square of the sine of the angle of yaw. We define the function $f(\ell)$ as

$$f(\ell) = (1 - \ell^2)\ (2H - 2V) - (\bar{\phi} - \ell)^2 \ . \tag{4.1}$$

This function dominates much of the motion of a shell in the same manner as the similar function dominates the theory of Abelian integrals. In fact if V is approximated by a truncated cosine series, f is a polynomial in ℓ, and the analysis depends strongly on the theory of elliptic and hyperelliptic integrals. Without the use of such sophisticated mathematics, much can still be learned about the overall motion from the roots of this polynomial.

Since the cosine of the angle of yaw must be a real quantity, it follows from (3.5) that the function f must be positive for any value of yaw the shell can assume consistently with the initial conditions. In particular, the yaw will oscillate between two neighboring roots ℓ_0 and ℓ_1 such that the function is positive between these two roots. There is of course the requirement that the cosine of a real angle must be of not more than unit magnitude. In this connection it is gratifying that the function is non-positive for ℓ of unit magitude.

The special case of V linear in ℓ corresponds to the common gravitational top. This case is discussed at length in reference 8 and, for that matter, most any text of classical mechanics.

The roots, ℓ_0 and ℓ_1, of f which bound the motion are often well adapted to use as secular (slowly varying) variables. Indeed the entire concept of stability hangs on these roots. In many cases it might be desirable to avoid the algebraic difficulties of finding roots of high order polynomial by carrying both the roots and H and $\emptyset$ as dependent variables.

5. THE ROOTS AS VARIABLES

In this section we shall do two things, find the differential equations for the roots and evaluate H and $\emptyset$ in terms of the roots. The first is necessary for using the roots as dependent variables and for determining the behavior of the variable Θ which will be introduced in the next section. The second is necessary if the roots are to be used exclusively as dependent variables.

We proceed as follows:

Let r be one of the roots of interest. Then

$$f(r) = (1-r^2)(2H-V(r)) - (\emptyset-r)^2 = 0$$

Differentiating this expression we obtain

$$(\partial f(r)/\partial r)r' = 2(\emptyset-r)\emptyset' - 2(1-r^2)H'$$

$$= 2(\emptyset-r) \left[-(\ell J_L + J_H k_2^{-2})(\emptyset-\ell)-(J_T k_1^{-2}-J_L)(1-\ell^2)\right]$$

$$-2(1-r^2) \left[-J_H k_2^{-2} (2H-2V(\ell)) - J_T k_1^{-2}(\emptyset-\ell) + J_L(1-\ell^2)/4S(\ell)\right].$$

Using $f(r) = 0$ and observing care in keeping ℓ and r separate, one obtains, by eliminating terms in $\emptyset^2$,

$$(\partial f(r)/\partial r)r' = 2J_L \Big\{-\ell(\emptyset-r)\ (r-\ell)+[2H-2V(r)]\ (r-\ell)(1+r\ell)$$

$$+\left[1/4S(r) - 1/4S\ (\ell)\right](1-\ell^2) + \tfrac{1}{2}\ (1-\ell^2)\ \frac{\partial f(r)}{\partial r}\Big\}$$

$$+ 2k_1^{-2}\ J_T\ (r-\ell)\Big\{1+r\ell - \emptyset(r+\ell)\Big\} \tag{5.1}$$

$$+ 2k_2^{-2}\ J_H \Big\{(r-\emptyset)(r-\ell) + 2(1-r^2)\left[V(r) - V(\ell)\right]\Big\}.$$

Each J is a function of ℓ.

It is certainly reasonable to assume that V and S are differentiable. In which case, using the mean value theorem, we can write:

$$(\partial f(r)/\partial r)r' = (1-\ell^2)\ J_L\ (\partial f\ (r)/\partial r) + (r-\ell)C\ (r,\ell)\ ,$$

Where

$$\lim_{\ell \to r} C(r,\ell) \text{ is finite.}$$

Anticipating the next section, we define

$$f(\ell) = (\ell_1-\ell)(\ell-\ell_0)\ F(\ell) \qquad 5.2$$

where F is bounded, positive, and non-vanishing in the closed interval (ℓ_0, ℓ_1). Then

$$(\partial f(\ell_0)/\partial r) = (\ell_1-\ell_0)\ F\ (\ell_0)$$

$$(\partial f(\ell_1)/\partial r) = (\ell_0-\ell_1)\ F\ (\ell_1)$$

and

$$\ell_0' = (1-\ell^2)\ J_L + \frac{\ell_0-\ell}{\ell_1-\ell_0}\ C\ (\ell_0,\ell)/F_0$$

$$\ell_1' = (1-\ell^2)\ J_L + \frac{\ell_1-\ell}{\ell_0-\ell_1}\ C(\ell_1,\ell)/F_1 \qquad 5.3$$

where F_0 and F_1 are the values of F evaluated at ℓ_o and ℓ_1.

A more detailed evaluation of the derivatives of the roots would depend on a knowledge of the functional form of V. For our purposes equations 5.3 are sufficient.

The next step, as indicated at the beginning of this section, is to find the functional relationship between the roots and H and $\bar{\emptyset}$. To do this most simply and to give the most understandable form, we introduce two new intermediate variables. These new variables are quite closely related to rather familiar concepts in exterior ballistics. In particular let δ_0 and δ_1 be the angles of yaw associated with ℓ_0 and ℓ_1. We then define two new variables

$$p = (1/2)\ (\delta_0 + \delta_1)$$

$$n = (1/2)\ (\delta_0 - \delta_1)\ . \qquad 5.4$$

The letters p and n are chosen for the connotation of precession and nutation. Here we are at variance with the majority of the literature in the field of exterior ballistics, but in keeping with the traditions of classical mechanics. That is in this paper we call nutation the motion involving changes in the angle of yaw and precession the average motion of the shell axis about the velocity vector. In this sense we shall see that nutation is always more nearly associated with the ballistic concept of fast motion while the precession can be either fast or slow.

From the two equations

$$f(\ell_0) = 0 \text{ and } f(\ell_1) = 0$$

eliminate H to obtain

$$\bar{\phi} = \frac{1 + \ell_0\ell_1}{\ell_0 + \ell_1} \pm \frac{\sqrt{(1-\ell_0^2)(1-\ell_1^2)}}{\ell_0 + \ell_1} \quad \sqrt{1 - 4\,\frac{\ell_0+\ell_1}{2}\,\frac{V_1-V_0}{\ell_1-\ell_0}}$$

where V_0 and V_1 are the values of V for ℓ_0 and ℓ_1.

Changing variables to p and n gives

$$\bar{\phi} = \frac{\cos n\ (1 \pm \sigma)}{2 \cos p} + \frac{\cos p\ (1 \mp \sigma)}{2 \cos n} , \qquad 5.5$$

where

$$\sigma = \sqrt{1 - 4 \cos p \cos n\ (V_1-V_0)/(\ell_1 - \ell_0)} , \qquad 5.6$$

which reduces to the usual σ in ballistics in those cases where V is linear in the cosine (K_m is constant). Now if ϕ is the azimuth angle as used in Reference 8,

$$\phi' = \frac{A\nu\ (\bar{\phi}-\ell)}{B\sin^2 \delta} \qquad 5.7$$

If we consider the case of pure precession (n = 0)

$$\phi' = \frac{A\nu\ (1 \pm \sigma)}{2B \cos p} \qquad 5.8$$

For the case of statically unstable shell ($0 < \sigma < 1$), there are two precession rates, both in the same direction, one fast and one slow. For the case of statically stable shell ($\sigma > 1$) there are also

two rates but now of different sign as well as magnitude. For the case of neutrally stable shell ($\sigma = 1$) there is only one rate and that is the vacuum rate as one would expect.

One final observation about the rates is of interest. For $p = 90^{\circ}$, there is only one possible precession rate. This motion is possible since the limit of σ as p approaches 90° is one. The slow motion is the possible one and in this case

$$\bar{\phi} = \cos^2 n \, (V_1 - V_0)/(\ell_1 - \ell_0)$$

In a similar manner as for equation 5.5 we obtain:

$$H = \frac{1}{2} \left\{ \tan^2 p \, \left(\frac{1 \pm \sigma}{2}\right)^2 + \tan^2 n \, \left(\frac{1 \pm \sigma}{2}\right)^2 \right\} + \frac{1}{2} \, (V_1 + V_o) \qquad 5.9$$

6. THE UNIFORMISING VARIABLE Θ

In Section 3 it is noted that ℓ is not a canonical coordinate. While time (actually arc length) is the coordinate conjugate to the Hamiltonian, it is more convenient to introduce another variable Θ by the relation

$$\ell = (1/2)(\ell_0 + \ell_1) + (1/2)(\ell_0 - \ell_1) \cos \Theta \quad . \qquad 6.1$$

Thomas calls this variable a uniformising variable since, if ℓ varies sinusoidally, Θ varies linearly. In any event, the transformation absorbs much of the oscillation.

Differentiating 6.1, substituting in 3.4, and rearranging terms gives

$$\frac{\ell_0 - \ell_1}{2} \sin \Theta \Theta' = \pm \frac{A\nu}{B} \sqrt{f(\ell)} - J_L(1-\ell^2) + \ell_0' \left(\frac{1+\cos \Theta}{2}\right) + \ell_1' \left(\frac{1-\cos \Theta}{2}\right)$$

Using 5.2 and 5.3 we obtain

$$\Theta' = \frac{A\nu}{B} \sqrt{F(\ell)} - \frac{\sin \Theta}{2(\ell_0 - \ell_1)} \left[\frac{C(\ell_0, \ell)}{F_0} + \frac{C(\ell_1, \ell)}{F_1} \right] \qquad 6.2$$

In general the presence of the $\ell_1 - \ell_0$ term in the denominator of the last term causes trouble when the roots are almost equal. A discussion of this case will be given in Section 8. Fortunately for roots

which are separated sufficiently that the manner of finding the average of ℓ is important, the entire term can be disregarded. This latter simplification is applicable only for the case of heavier-than-air projectiles (see below). Since this is the case with which we are concerned, we shall use

$$\theta' = \frac{A\nu}{B} \sqrt{F(\ell)} \qquad 6.3$$

as the equation of the fundamental motion. This is similar to the use of the approximation $Q_1 = t$ in Section 1.

The frequency of nutation can be defined as the reciprocal of the time (arc length) in which θ changes by 2π. We call this time (arc length) the period, P, where

$$P = \int_{\theta_0}^{\theta_0+2\pi} \frac{d\,\theta}{\frac{A\nu}{B}\sqrt{F}} \qquad 6.4$$

To establish under what conditions equation 6.3 is valid, we shall evaluate the C and F terms for $\ell_0 = \ell_1 = \ell = \cos p$. We shall then consider under what conditions the term $A\nu\sqrt{F(\cos p)}/B$ dominates the term $C(\cos p, \cos p)/F(\cos p)\left[\ell_0 - \ell_1\right]$.
Now, using 5.1, 5.5, and 5.9, we obtain

$$\begin{aligned} C(\cos p, \cos p) &= 2J_L \left[\tan^2 p\,(1\pm\sigma)^2/4 - \sin^2 p\,(1-\sigma^2)/4\right] \\ &\mp 2J_T\, k_1^{-2} \sin^2 p\, \sigma \\ &\mp 2J_H\, k_2^{-2} \tan p \sin p\,(1\pm\sigma)\ /2 \ . \end{aligned} \qquad 6.5$$

Equation 6.5 may be used in detail to obtain an estimate of the size of C. However, in the cases which are usually of interest, we can merely say that C is of the order of magnitude of $(\rho d^3/m)\sin^2 p$, and hence $C/\left[\ell_1 - \ell_0\right]$ is of the order of magnitude of $(\rho d^3/m)(\sin p)/(\sin n)$. The comparison is then between $A\nu\sqrt{F}/B$ and $(\rho d^3/m)(\sin p)/(\sin n)F$. That is, the approximation 6.3 is invalid for n such that

$$\sin n < (\rho d^3/m) \sin p \,/\, \left[(A/B)\, F^{3/2}\right] . \qquad 6.6$$

For spin stabilized shell we may use $F \cong 1$ and $K_M k_2^{-2} \cong 1$ to get as an order of magnitude approximation

$$\sin n < (1/4S)(A\nu/B) \quad \sin p \qquad 6.7$$

for the approximation 6.3 to be invalid.

A more specific analysis along the same lines is necessary for any particular fin-stabilized shell.

7. THE FUNCTION F

The function F, together with $A\nu/B$, determines the frequency of the nutation of the shell (6.4). If V is at most quadratic in ℓ, the nutation can be expressed in terms of elliptic integrals. If V is of higher order. the solution requires hyper-elliptic functions (or series expansions). It is instructive to reduce F to a form similar to the expressions for the nutational frequency found in the usual ballistic theory. (In view of our definition of nutation, the nutational frequency we shall obtain will be equal to the difference of the two frequencies of the usual ballistic theory).

We first note that if

$$f(\ell_0) = f(\ell_1) = 0$$

Then, by using the mean value theorem,

$$f(\ell) = -(1/2)(\ell_0-\ell)(\ell-\ell_1) \frac{d^2 f(\bar{\ell})}{d\ell^2} \qquad 7.1$$

where $\bar{\ell}$ lies between ℓ_0 and ℓ_1 and depends on ℓ.

Letting

$$\bar{V} = V(\bar{\ell}) \quad \text{etc.}$$

we have

$$F = 1 + (2H-2\bar{V}) - 4\bar{\ell} \frac{d\bar{V}}{d\ell} + (1-\ell^2) \frac{d^2\bar{V}}{d\ell^2} \quad . \qquad 7.2$$

Several cases are simple enough to be of interest. First consider the case of small nutation. Letting n go to zero and using 7.2, 5.9, and 5.6, we get the following expression for small nutation.

$$F = \sigma^2 + \tan^2 p \left(\frac{1 \pm \sigma}{2}\right)^2 + \sin^2 p \frac{d^2 V}{d\ell^2} \qquad 7.3$$

The other special case we shall consider is that for V quadratic in ℓ. In particular let $V = a\ell + b\ell^2$. Then

$$F = \sigma^2 + \tan^2 p \left(\frac{1 \pm \sigma}{2}\right)^2 + \tan^2 n \left(\frac{1 \pm \sigma}{2}\right)^2$$
$$+ \frac{1 - \sigma^2}{2} \left(1 - \frac{2\ell}{\ell_1 + \ell_0}\right) + 2(1 - \ell^2)\ b \qquad 7.4$$

8. THE BEHAVIOR FOR SMALL NUTATION

In developing the equations for the uniformising variable, mention was made that the right hand side of equation 6.2 could cause trouble. Such is often the case when the nutation is quite small (while a specific check would be required for each case, it appears that generally the approximation presented by 6.3 is valid for any nutation which can be considered sensibly different from zero. This appears to be the case as long as the density of the projectile is large compared to the density of the resisting medium.) While the case of extremely small nutation presents no problem as to the choice of a proper averaging technique, for completeness it is desirable to discuss this case. Indeed, in one sense at least, it is mandatory that an explanation be given as to why a perturbation technique appears to break down under those conditions for which the perturbed variables should be changing the least. It is desirable to show that the difficulties are only in the coordinate system.

Actually the variables H and $\bar{\phi}$ do change slowly at all times. The difficulty is in the roots of f. When the two roots are close together, very small changes in the parameters of the polynomial cause proportionally much larger changes in the roots. What happens, as we see below, is that in certain cases the uniformising variable has limited variation, and the roots oscillate markedly.

For the case of small nutation equations 5.3 and 6.2 can be approximated by

$$\Theta' = \frac{A\nu \sqrt{F}}{B} - \frac{\sin \Theta}{(\ell_0 - \ell_1)} \quad C/F$$
$$(\ell_0 - \ell_1)' = \cos \Theta \quad C/F \qquad 8.1$$

where C, F, and σ are evaluated for $\ell_0 = \ell_1$.

These equations can be normalized by introducing the variables

$$x = \frac{(\ell_0 - \ell_1) \quad CA\nu \qquad \sqrt{F}}{FB}$$

$$dz = \frac{A\nu \sqrt{F}\, ds}{B} \quad . \qquad 8.2$$

The normalized equations are

$$\frac{d\theta}{dz} = 1 - \frac{\sin \theta}{x}$$

$$\frac{dx}{dz} = \cos \theta \quad . \qquad 8.3$$

A graphical representation of these equations appears in Figure 1. It can be seen that two types of solutions are possible: closed paths around the points $x = 1$, $\theta = (2k + \frac{1}{2})\pi$ or around the points $x = -1$, $\theta = (2k + \frac{3}{2})\pi$; and open paths in which θ increases indefinitely. Actually the variables x and θ are not the observable quantities. What is observable is the change in the cosine of the yaw, this change being proportional to $x \cos \theta$. If we denote this quantity by y then 8.1 gives

$$\frac{d^2y}{dz^2} + y = 0$$

which by the definition of z means that the cosine of the yaw has the usual frequency, viz. $A\nu \sqrt{F}/B$.

9. THE SECULAR EQUATIONS

While equations 6.3 and the first two of equations 3.4 could be used their present form for machine computation, the presence of high frequency oscillations is a source of possible divergence of the truncation error in the computational techniques unless extremely small integration intervals are employed[12]. Further the presence of three variables upon which the equations depend explicitly, rules out the possibility of a geometric interpretation with any intuitive value. A simplification of the equations is therefore indicated from both the analytic and computational views.

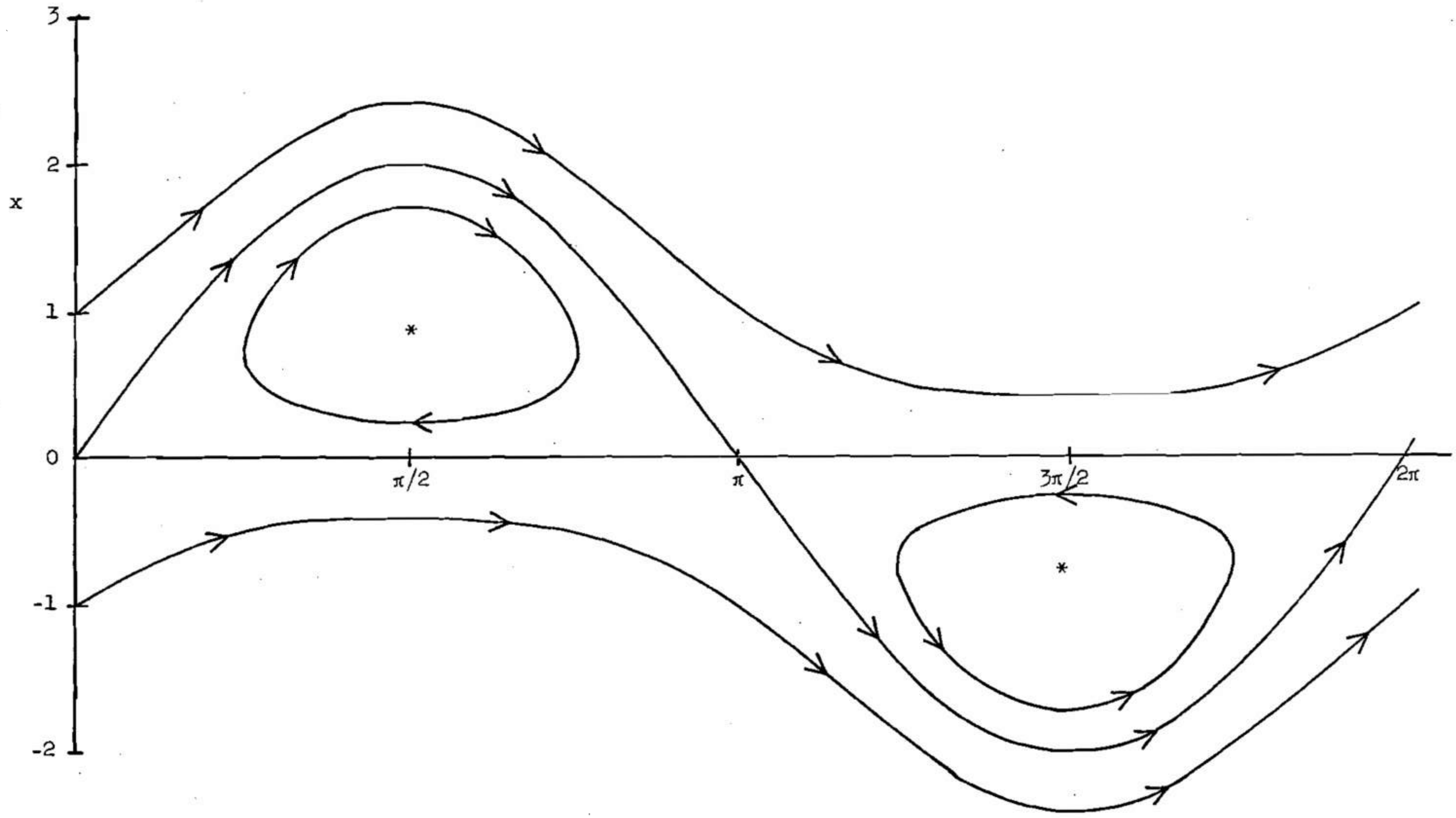


Figure 1

To this effect we invoke the same approximation which was used in obtaining equation 6.3 from 6.2, namely, the transient times are large compared to the nutational period. The secular equations (see 1.6) are obtained by averaging the dependence of the first two of equations 3.4 on Θ over a period of nutation. It is assumed that over this period H and $\bar{\phi}$ are constant. The average will depend upon the functional form and roots (ℓ_0 and ℓ_1) of f.

Somewhat more specifically the modus operandi is as follows:

Equations 3.4 may be written as

$$H' = a_{11} H + a_{12} \bar{\phi} + a_{13}$$

$$\bar{\phi}' = a_{22} \bar{\phi} + a_{23} \qquad 9.1$$

where the a's are functions of ℓ and hence of Θ and the roots of f. The a's are replaced by their averages

$$\bar{a} (\ell_0, \ell_1) = \frac{1}{P} \int_s^{s+P} a(\ell_0, \ell_1, \Theta) ds$$

$$= \frac{1}{P} \int_0^{2\pi} \frac{a(\ell_0, \ell_1, \Theta) d\Theta}{\frac{Av}{B} \sqrt{F(\ell_0, \ell_1, \Theta)}} \qquad 9.2$$

$$P = \int_0^{2\pi} \frac{d\Theta}{\frac{Av}{B} \sqrt{F(\ell_0, \ell_1, \Theta)}}$$

The roots (ℓ_0 and ℓ_1) are considered constant in these integrations.

Usually the a's can be written as polynomials in ℓ and hence as

$$a_{ij} (\ell) = a_{ijk} (\ell_0, \ell_1) \cos^k \Theta \qquad 9.3$$

with summation over k.

Therefore

$$\bar{a}_{ij} = a_{ijk}\, b_k$$

where

$$b_k = \frac{\displaystyle\int_0^{2\pi} \frac{\cos^k\theta \, d\theta}{\sqrt{F}}}{\displaystyle\int_0^{2\pi} \frac{d\theta}{\sqrt{F}}} \qquad 9.4$$

Evaluation of the latter quantity involves complete elliptic integrals of the first and second kind when f is cubic (see next section), complete elliptic integrals of the first, second, and third kind when f is quartic and complete hyperelliptic integrals when f is of higher order.

The final secular equations are

$$\begin{aligned} H' &= a_{11k} b_k H + \bar{\phi}\, a_{12k} b_k + a_{13k} b_k \\ \bar{\phi}' &= a_{22k} b_k \bar{\phi} + a_{23k} b_k \end{aligned} \qquad 9.5$$

where the a's and b's are functions of ℓ_0 and ℓ_1.

We may now think of the dynamic history of the shell as being a path in the H, $\bar{\phi}$ plane. A shell may be defined as dynamically stable in an H, $\bar{\phi}$ region if all paths from that region lead to the H, $\bar{\phi}$ point corresponding to $\ell_0 = \ell_1 = 1$. The region of physically consistent H and $\bar{\phi}$ (that is H and $\bar{\phi}$ such that f is positive for some values of ℓ in the interval $-1 < \ell < 1$) is bounded by the curve of values of H and $\bar{\phi}$ for which the roots are equal (n=0). Any limit points on this curve represent stable pure precessional motion. Limit points inside the region represent stable "elicyclic" motion. In those cases where the curves for $H' = 0$ and for $\bar{\phi}' = 0$ can be obtained, the intersections of these curves (singular points) may be analysed and an overall description of the motion obtained (in the same manner as reference 7). The latter situation is not too likely since, even if the elliptic integrals can be approximated, a system of three nonlinear algebraic equations must be considered (two for the vanishing of the derivatives and one for the

roots of f). A more profitable line of attack would probably be to lay down a sufficiently dense set of numerical solutions for reliable inferences to be drawn about the general motion.

However, there are many special cases in which such things as restricted initial conditions, etc. allow analytic approximations to be made and a somewhat closed solution to be obtained.

10. THE CASE OF A CUBIC f

For the case in which the stability factor (S) is a constant, (V is linear in ℓ as in the usual gravitational top), the function f is a cubic polynomial in ℓ, and the secular equations can be written in terms of complete elliptic integrals of the first and second kind. In fact, only one transcendental function need be used, namely E/K.

To demonstrate this fact and to derive the detailed equations for the cubic case, it is convenient to use all three of equations 3.4. Averaging with respect to ℓ is equivalent to averaging with respect to the uniformising variable θ.

$$\overline{\ell^n} = (1/P) \int_{\ell_1}^{\ell_0} \frac{\ell^n}{\sqrt{f}} \, d\ell \qquad 10.1$$

$$P = \int_{\ell_1}^{\ell_0} \frac{d\ell}{\sqrt{f}}$$

where again the roots of f are assumed to remain constant during the integration.

Now

$$\begin{aligned} f &= (1 - \ell^2)(2H - \ell/2S) - (\bar{\phi}-\ell)^2 \\ &= (1/2S)\ (\ell - \ell_0)(\ell - \ell_1)(\ell - \ell_2) \\ \ell_2 &= 2S(2H + 1) - \ell_0 - \ell_1 \end{aligned} \qquad 10.2$$

Noting that f(1) and f(-1) are negative and that since f is an odd order polynomial it must approach plus infinity either on the right ($S > 0$) or on the left ($S < 0$), we conclude that there is either a root

$\ell_2 > 1$, $S > 0$ or a root $\ell_2 < -1$, $S < 0$. We shall consider the case for $S > 0$ (the moment of force tends to overturn the shell). The other case is quite similar.

Since we are interested in the ratio of two integrals, we may neglect the factor of $1/2S$. Then

$$\overline{\ell^n} = \frac{\displaystyle\int_{\ell_1}^{\ell_0} \frac{\ell^n \, d\ell}{\sqrt{(\ell - \ell_0)(\ell-\ell_1)(\ell-\ell_2)}}}{\displaystyle\int_{\ell_1}^{\ell_0} \frac{d\ell}{\sqrt{(\ell-\ell_0)(\ell-\ell_1)(\ell-\ell_2)}}} \qquad 10.3$$

where

$$\ell_2 > 1 > \ell_0 > \ell_1$$

Making the substitution

$$\begin{aligned} sn^2 u &= (\ell-\ell_1)/(\ell_0-\ell_1) \\ k^2 &= (\ell_0-\ell_1)/(\ell_2 \ell_1) \end{aligned} \qquad 10.4$$

we get

$$\begin{aligned} \overline{\ell^n} &= (1/K) \int_0^K (\ell_1 + (\ell_0-\ell_1)\, sn^2 u)^n \, du \\ &= (1/K)\, \ell_1^{n-j} (\ell_0-\ell_1)^j \binom{n}{j} A_{2j} \end{aligned} \qquad 10.5$$

summing on j and where

$$A_{2j} = \int_0^K sn^{2j} u \;\; du \quad .$$

Now (see reference 13)

$$\begin{aligned} A_0 &= K \\ A_2 &= k^{-2} (E - K) \\ A_{2(j+1)} &= \frac{2j(1+k^2)\, A_{2j} + (1-2j)\, A_{2(j-1)}}{(2j+1)\, k^2} \end{aligned} \qquad 10.6$$

Hence, since there integrals are divided by K to form the averages, $\overline{\ell^n}$ is a function of ℓ_0, ℓ_1, k^2, and E/K.

Finally we note that, if we write for the a's in equation 9.1

$$a_{ij} = c_{ijn} \ell^n$$

summing on n where the c's are constants,

$$\bar{a}_{ij} = c_{ijn} \overline{\ell^n} \quad , \tag{10.7}$$

and we have the secular equations a functions of H, ϕ, ℓ_0, and ℓ_1 with E/K as the only transcendental function.

One further item is of interest in connection with a cubic f. That is the geography of the H, $\bar{\phi}$ plane (Figure 2). The curve which bounds the region of permissible motion has three parts: pure slow precession, pure fast precession with p less than 90°, and pure fast precession with p greater than 90°. The lines for one root = 1(0°) and for one root = -1 (180°) are drawn and are seen not to intersect (the motion cannot oscillate between 0 and 180° unless the spin is zero). One other line is included. That is the line for $H = V(\ell_0)$, $\bar{\phi} = \ell_0$. Points on this line are those conditions which correspond to the case of an initially stationary shell axis. These are the initial conditions for shell fired from aircraft.

11. A ONE-DIMENSIONAL EXAMPLE

The following is given to illustrate the techniques described thus far. Consider the one-dimensional equation

$$x'' + (a + b\,x^2)x' + x^3 = 0.$$

We need only define one secular function, namely

$$H = (1/2)(x')^2 + (1/4)(x^4) \quad ,$$

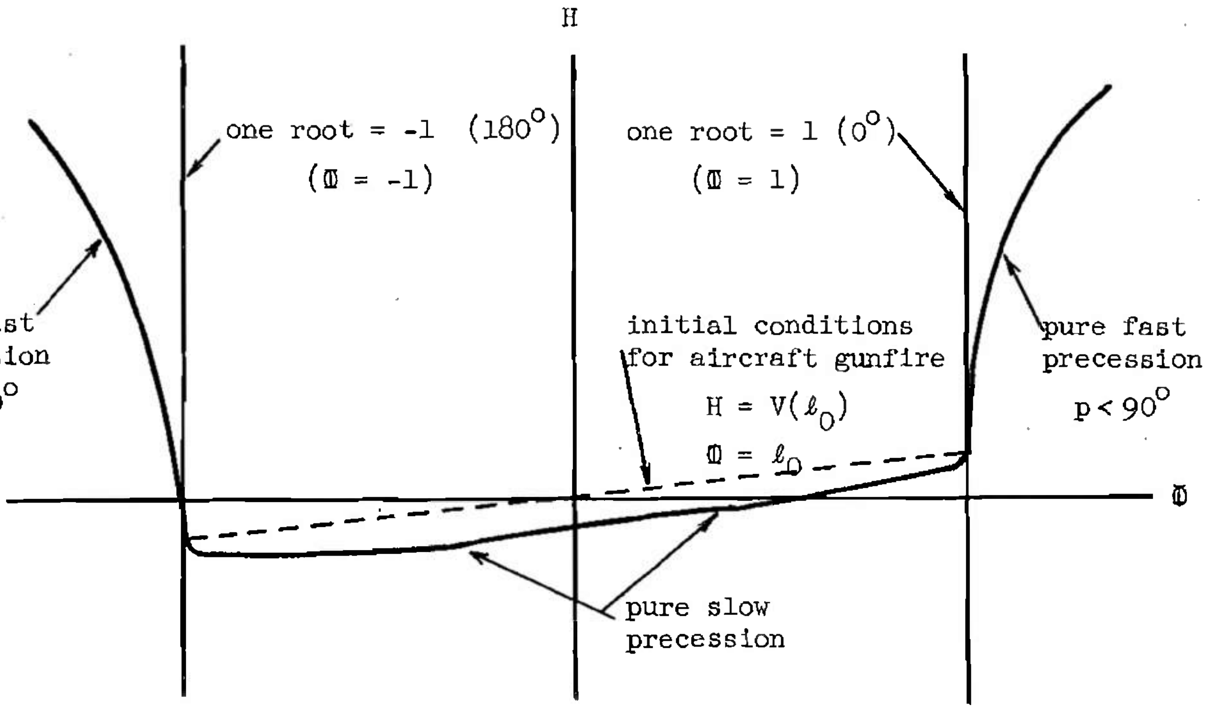
which is, as usual, the sum of the kinetic and the potential energy. Then

$$H' = -\,(a + bx^2)(H - (1/2)x^4)$$

$$x' = \sqrt{2H - (1/2)x^4}$$

Define two new variables r and Θ by

$$x = r\cos\Theta$$
$$r^4 = 4H \quad .$$



H, Φ PLANE

Figure 2

Then

$$\Theta' = r\sqrt{(1+\cos^2\Theta)/2} - (a + br^2\cos^2\Theta)(1+\cos^2\Theta)\sin\Theta\cos\Theta$$

$$r' = -(1/2)(a + b\,r^2\cos^2\Theta)(1-\cos^4\Theta)r \quad .$$

This is an interesting case. If r is small we cannot ignore the last term in the Θ' equation. Indeed for small r the motion becomes over-damped. However, we shall start with a value of r which is large enough to allow dropping the second term in the Θ' equation. This is on the assumption that $a + br^2$ is smaller than $r/\sqrt{2}$

With this assumption, we let

$$\cos\Theta = \mathrm{cn}(u) \qquad (\mathrm{cn} = \text{cosine amplitude})$$

in which case the secularized r' equation becomes

$$r' = -(r/2K)\int_0^K (a + br^2\mathrm{cn}^2u\ x - a\mathrm{cn}^4u - b\mathrm{cn}^6u)du$$

$$= -r\left[a/3 + b(2E/K - 1)/5\right] \quad .$$

GEDA solutions were run for two specific cases (Figures 3 and 4). The oscillating curve is the actual solution. The envelopes are solutions of the secular equation and give the amplitude of the oscillation. The agreement is good.

12. THE MURPHY NONLINEAR TREATMENT

C. H. Murphy[7] has been quite successful in applying the methods of nonlinear vibration theory to the case of shell motion. It is desirable to tie his methods to the methods of Hamiltonian mechanics. To do so, we shall consider a somewhat different form of the equations than that of equations 2.1, and for simplicity we shall consider nonlinearity in the Magnus term alone.

We take as the fundamental equation of motion (after Murphy)

$$\lambda'' + (\mathscr{H} - i\bar{\nu})\ \lambda' + (-M - i\nu T)\ \lambda = 0 \qquad 11.1$$

with $\bar{\nu} = (A/B)\nu$

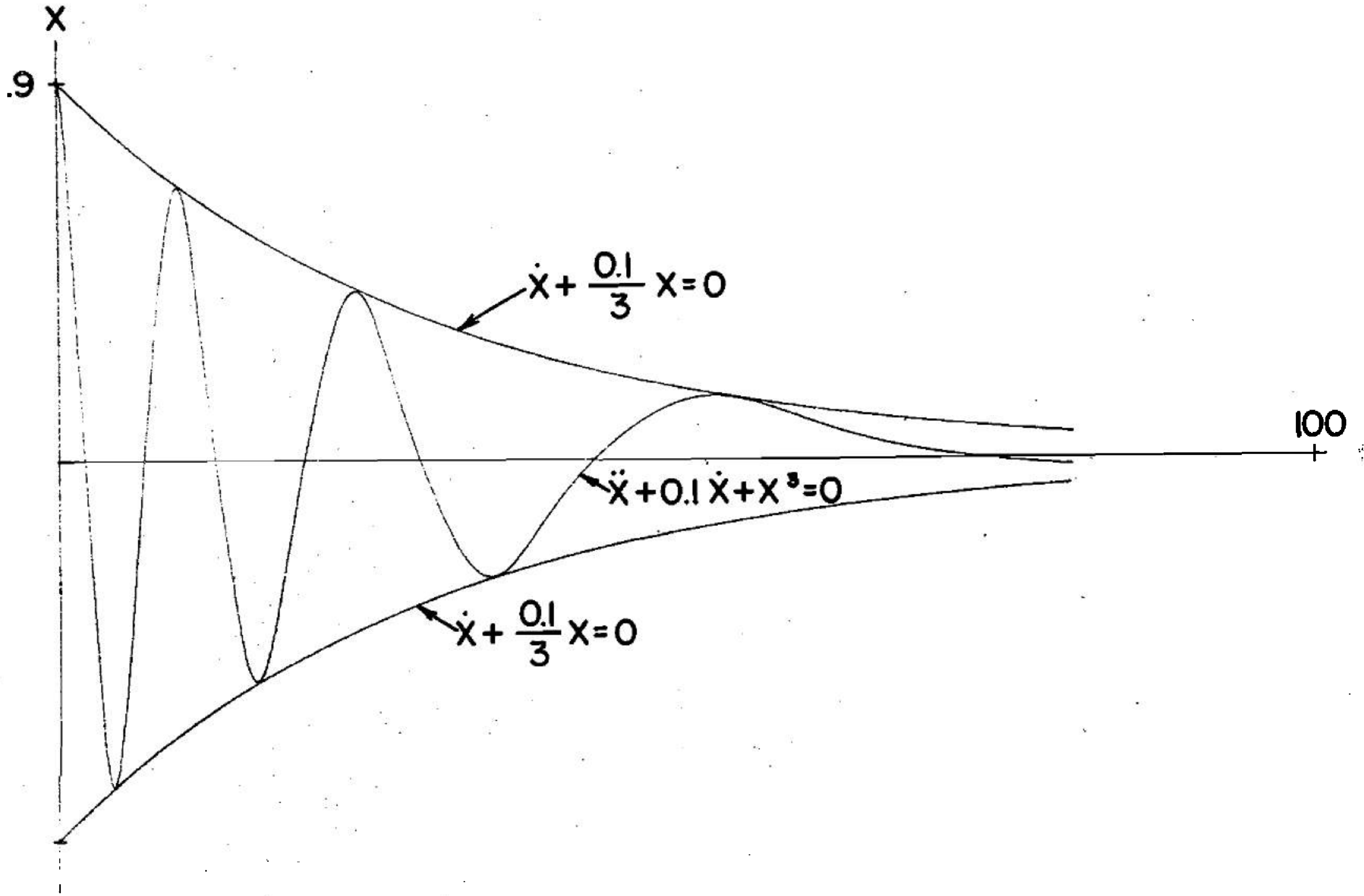


FIGURE 3

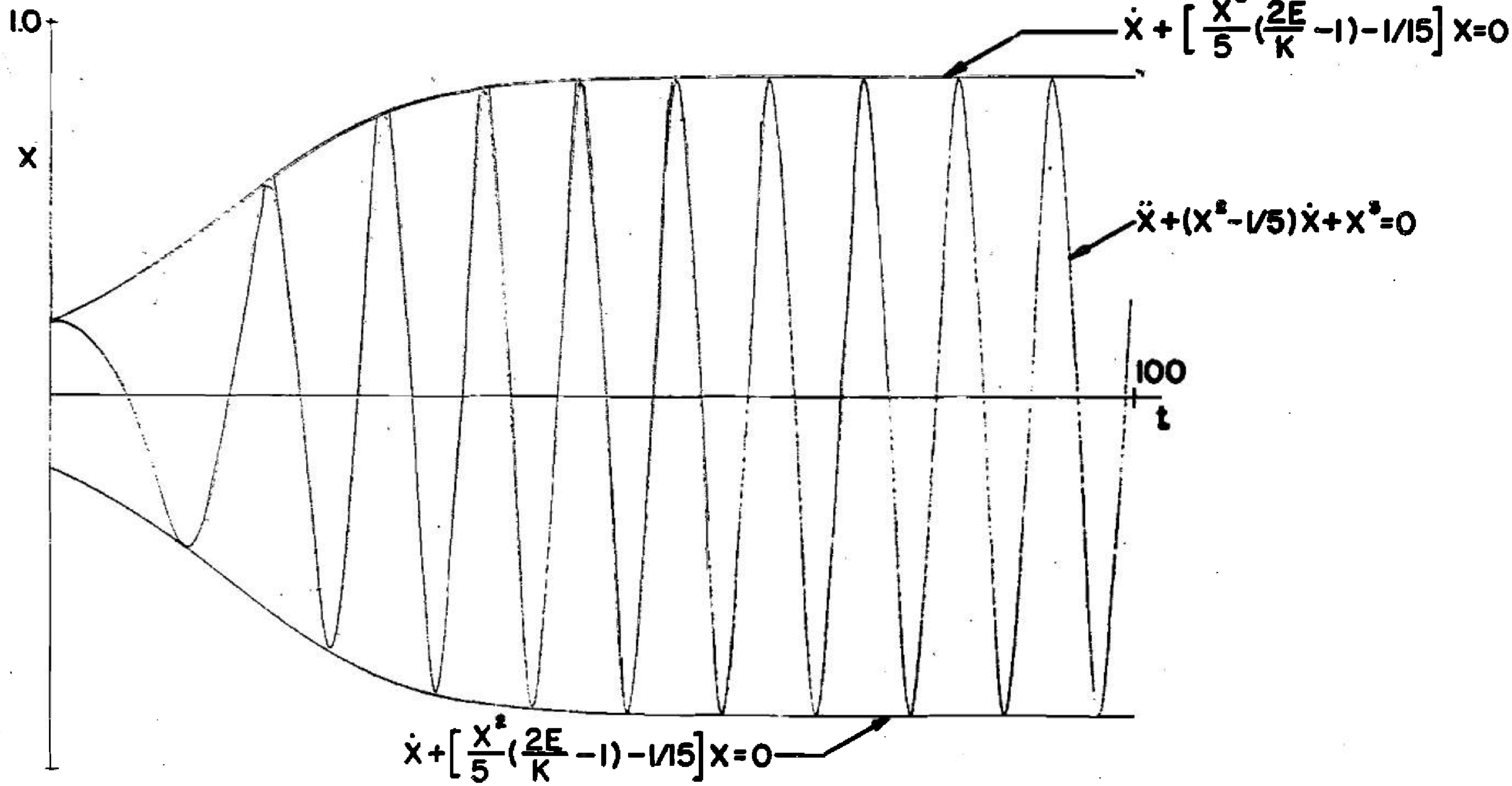

$\dot{X} + \left[\frac{X^2}{5}\left(\frac{2E}{K} - 1\right) - 1/15 \right] X = 0$
1.0
X
$\ddot{X} + (X^2 - 1/5)\dot{X} + X^3 = 0$
100
t
$\dot{X} + \left[\frac{X^2}{5}\left(\frac{2E}{K} - 1\right) - 1/15 \right] X = 0$

FIGURE 4

$$\mathcal{K} = J_L + k_2^{-2} J_H$$

$$M = k_2^{-2} J_M$$

$$T = J_L - k_1^{-2} J_T = T_1 + T_2 \quad \lambda\bar{\lambda}$$

In these equations we have neglected the geometric nonlinearities. These could be added, but the algebra would become more difficult.

We now define the variables

$$H = (1/2)\ (\lambda' \bar{\lambda}' - M\lambda\bar{\lambda})\ \sqrt{\bar{\nu}}^2$$

$$\phi = (\bar{\lambda}\lambda' - \lambda\bar{\lambda}')/2i\bar{\nu} - \lambda\bar{\lambda}/2 \qquad 11.2$$

These variables are closely related to our original definitions. One difference should be noted, namely, $\bar{\phi}_{new} = \bar{\phi}_{old} - 1$. This means that a negative $\bar{\phi}$ is associated with the slow precession, and a positive $\bar{\phi}$, with fast precession.

The differential equations in terms of the new variables are

$$\bar{\phi}' = -\mathcal{K}(\phi + \lambda\bar{\lambda}/2) + T\lambda\bar{\lambda}$$

$$H' = -2\mathcal{K}(H + M\lambda\bar{\lambda}/2\bar{\nu}^2) + T\left[\phi + \lambda\bar{\lambda}/2\right] \qquad 11.3$$

$$(\lambda\bar{\lambda})' = (\sin^2\delta)' = \pm\ 2\bar{\nu}\sqrt{2\sin^2\delta\ (H + \sin^2\delta/8S) - (\phi + (\sin^2\delta)/2)^2}$$

$$= \pm\ \bar{\nu}\sigma\ \sqrt{(\sin^2\delta_1 - \sin^2\delta)(\sin^2\delta - \sin^2\delta_0)}$$

Let

$$(K_1 + K_2)^2 = \sin^2\delta_1$$

$$(K_1 - K_2)^2 = \sin^2\delta_0 \qquad 12.4$$

Then

$$\sin^2\delta = K_1^2 + K_2^2 + 2K_1K_2 \cos\Theta \qquad 12.5$$

and

$$K_{1,2}^2 = \frac{2H - (1 \mp \sigma)\ \bar{\phi}}{\sigma^2} \qquad 12.6$$

Using 12.5 and 12.6 in 12.2 and taking averages with respect to θ, by using

$$(1/2\pi) \int_0^{2\pi} (K_1^2 + K_2^2 + 2K_1K_2 \cos \theta)\, d\theta = K_1^2 + K_2^2 \qquad 12.7$$

$$(1/2\pi) \int_0^{2\pi} (K_1^2 + K_2^2 + 2K_1K_2 \cos \theta)^2\, d\theta = K_1^4 + K_2^4 + 4K_1^2K_2^2 \quad ,$$

we get

$$(K_{1,2}^2)' = 2K_{1,2}^2 \frac{-\mathcal{H}\phi_{1,2}' + \bar{\nu}\left[T_1 + T_2\,(K_{1,2}^2 + 2K_{2,1}^2)\right]}{\phi_{1,2}' - \phi_{2,1}'} \qquad 12.8$$

where

$$\phi_{1,2}' = \bar{\nu}(1 \pm \sigma)/2 \quad .$$

This is Murphy's result.

13. AIRCRAFT GUNFIRE

In bomber defense gunfire at high altitude, stability factors are quite high, and the initial conditions are such that there is a large slow precession and practically no nutation. Two things are of interest: the damping of the precession when the nutation is zero and the stability of small nutation. We assume the stability factor is infinite which allows us to approximate 5.5 and 5.9 by

$$\bar{\phi} \cong (\cos p)/\cos n \qquad 13.1$$

$$H \cong (1/2)\, n^2$$

putting 12.1 into 3.4 using the relation

$$\cos \delta = \cos p \cos n + \sin p \sin n \cos \theta \ ,$$

and keeping no terms in n in the p' equation and only linear terms in n in the n' equation, we get

$$p' = -(J_L - J_T\, k_1^{-2}) \sin p \qquad 13.2$$

$$n' = -J_H\, k_2^{-2}\, n - J_T\, k_1^{-2}\, (n \cos p - \sin p \cos \theta) \quad .$$

The first equation has been checked against complete integrations of the dynamical problem, was found to give good agreement, and is being used in the computation of aircraft firing tables.

To be useful the second equation must be secularized by averaging the Θ term. The final equation will be in the form

$$n' = -J_H k_2^{-2} n - J_T k_1^{-2} n \cos p + (1/2\pi) \sin p \, K_1^{-2} \int_0^{2\pi} J_T \cos\Theta \, d\Theta$$

To evaluate the last term let

$$J_T = \sum_{m=0} c_m (\cos p \cos n + \sin p \sin n \cos\Theta)^m$$

$$= \sum c_m (\cos^m p + m \cos^{m-1} p \sin p \cos\Theta \, n)$$

Then

$$(1/2\pi) \int_0^{2\pi} J_T \cos \Theta \, d\Theta = \frac{n}{2} \sum m \, c_m \cos^{m-1} p \sin p \qquad 13.4$$

14. NON-RIGID SHELL

The Hamiltonian formulation is well adapted to the consideration of compound shell. In many cases there are limitations imposed on some of the degrees of freedom of the component parts. We shall consider one special case, that of a small mass which is constrained to move in a plane perpendicular to the shell axis and subject to a linear restoring force centered on the shell axis. In linearizing the problem we shall make extravagant use of canonical transformations in the hope that the repetition will engender a feeling for canonical transformations.

Consider the situation pictured in Figure 5. The y axes are space fixed axes. The x axes are fixed in the main shell body (the x_3 axis is the axis of symmetry of the main shell body). The ζ_1 and ζ_2 axes define the plane in which the small mass may move. The ζ_1 axis is taken in the plane defined by x_3, y_3. The origin of the ζ axes is at a distance a from the center of gravity of the main shell (origin of the x and y axes).

The main shell body is acted upon by a moment whose potential energy is $\mu \cos \Theta$. This is again the assumption that V is linear in ℓ. The spring has a potential energy of $(1/2)K(\zeta_1^2 + \zeta_2^2)$. The velocity components of the small mass (m) are given by

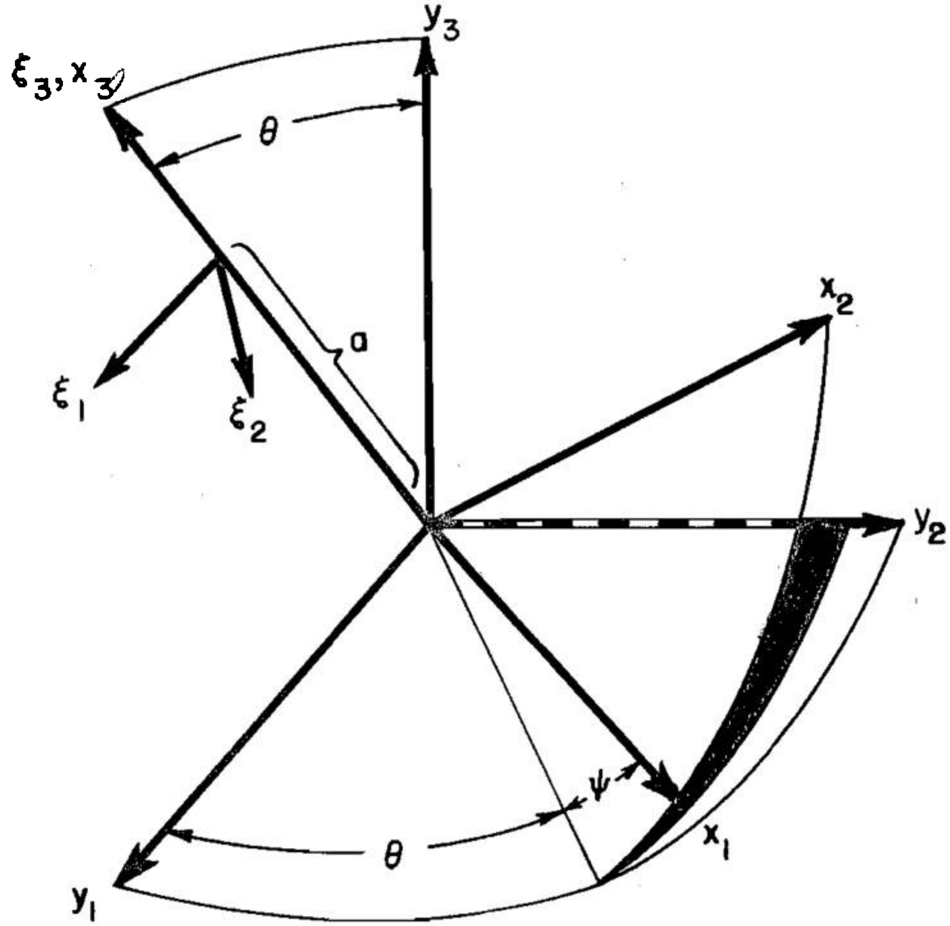


FIGURE 5

$$v_1 = a\dot{\theta} + \dot{\zeta}_1 - \dot{\phi}\ \cos\theta\ \zeta_2$$

$$v_2 = \dot{\phi}\ (a\ \sin\theta + \zeta_1\ \cos\theta) + \dot{\zeta}_2 \qquad 14.1$$

$$v_3 = -\dot{\phi}\ \ \zeta_2 \sin\theta - \dot{\theta}\ \zeta_1$$

The Hamiltonian for the system is

$$H = (B/2)(\dot{\theta}^2 + \dot{\phi}^2\ \sin^2\theta) + (A/2)\ (\dot{\psi} + \dot{\phi}\ \cos\theta)^2 + \mu\ \cos\theta$$

$$+ (m/2)(v_1^{\ 2} + v_2^{\ 2} + v_3^{\ 2}) + (K/2)(\zeta_1^{\ 2} + \zeta_2^{\ 2}) \ , \qquad 14.2$$

where A is the axial moment of inertia of the main shell body and B the transverse moment of inertia.

The five momenta conjugate to the coordinates ζ_1, ζ_2, ψ, θ, and ϕ are respectively:

$$p_1 = mv_1$$

$$p_2 = mv_2$$

$$p\psi = A(\dot{\psi} + \dot{\phi}\ \cos\theta) \qquad 14.3$$

$$p_\theta = B\dot{\theta} + ap_1 + m\zeta_1^{\ 2}\ \dot{\theta} + m\zeta_1\zeta_2\ \sin\theta\ \dot{\phi}$$

$$p_\phi = B\ \sin^2\theta\dot{\phi} + p\psi\cos\theta + \zeta_2^{\ 2}\ \sin^2\theta\ m\ \dot{\phi}\ \zeta_1\zeta_2 m\ \sin\theta$$

$$- \zeta_2\ \cos\theta p_1 + (a\ \sin\theta + \zeta_1\ \cos\theta)\ p_2$$

We shall make repeated use of canonical transformations with a generating function of the form $F(q_{old}, p_{new})$. The transformation equations for this function are

$$q_{i_{new}} = \frac{\partial F}{\partial p_{i_{new}}}$$

$$p_{i_{old}} = \frac{\partial F}{\partial q_{i_{old}}}$$

As a simplification in the notation only those conjugate pairs of variables actually changed by the transformation will be written in the generating fundtion and in the result. Those variables not included will be assumed to be subject to the identity transformation.

For our first transformation we take as the generating function

$$F = \phi p_3 + (\psi + \phi) p_4 \quad , \tag{14.4}$$

which gives us two new coordinates and momenta

$$\begin{aligned} q_3 &= \phi \\ q_4 &= \psi + \phi \\ p_3 &= p_\phi - p_\psi \\ p_4 &= p_\psi \end{aligned} \tag{14.5}$$

Essentially this transformation removes the spin from the momentum p_ϕ.

We now define a second transformation with generating function

$$\begin{aligned} F = \sin\theta \sin q_3\, p_5 - \sin\theta \cos q_3\, p_6 + (\zeta_1 \sin q_3 + \zeta_2 \cos q_3) p_7 \\ + (-\zeta_1 \cos q_3 + \zeta_2 \sin q_3)\, p_8 \end{aligned} \tag{14.6}$$

which gives the transformation equations

$$\begin{aligned} p_\theta &= \cos\theta \sin q_3\, p_5 - \cos\theta \cos q_3\, p_6 \\ p_3 &= \sin\theta \cos q_3\, p_5 + \sin\theta \sin q_3\, p_6 + (\zeta_1 \cos q_3 - \zeta_2 \sin q_3)\, p_7 \\ &\qquad + (\zeta_1 \sin q_3 + \zeta_2 \cos q_3)\, p_8 \\ p_1 &= p_7 \sin q_3 - p_8 \cos q_3 \\ p_2 &= p_7 \cos q_3 + p_8 \sin q_3 \\ q_5 &= \sin\theta \sin q_3 \\ q_6 &= -\sin\theta \cos q_3 \\ q_7 &= \zeta_1 \sin q_3 + \zeta_2 \cos q_3 \\ q_8 &= -\zeta_1 \cos q_3 + \zeta_2 \sin q_3 \end{aligned} \tag{14.7}$$

This transformation introduces the direction cosines of the shell axis (q_5 and q_6) and the direction cosines of the vector between the center of force and the small mass (q_7 and q_8). These direction cosines are for the y_1, y_2 axes.

To simplify the situation, we shall consider q_{5}, 6, 7, 8, their derivatives, and m to be small. We shall let $K = mf^2$ and assume that the frequency of the spring (f) does not vanish as $m \to 0$. Note that this makes p_7 and p_8 second order terms. Further we shall keep only up to third order terms in the Hamiltonian. To do this note that to second order terms

$$B\dot{\Theta} = (p_5 - ap_7)\sin q_3 - (p_6 - ap_8)\cos q_3 \qquad 14.8$$

$$B\sin\Theta\,\dot{\phi} = p_4\frac{\sin\Theta}{2} + (p_5 - ap_7)\cos q_3 + (p_6 - ap_8)\sin q_3 .$$

Therefore, to third order terms

$$H = (1/2B)\left\{(p_5 - ap_7)^2 + (p_6 - ap_8)^2 + p_4\left[-(p_5 - ap_7)q_6 + (p_6 - ap_8)q_5\right]\right.$$
$$\left. + p_4^2\frac{q_5^2 + q_6^2}{4}\right\} + \frac{1}{2A}p_4^2 - \frac{\mu}{2}\,q_5^2 + q_7^2 + \frac{1}{2m}(p_7^2 + p_8^2) + \frac{mf^2}{2}(q_7^2 + q_8^2) \qquad 14.9$$

For a final transformation we shall use the generating function

$$F = q_5 p_9 + q_6 p_{10} + (aq_5 + q_7)p_{11} + (aq_6 + q_8)p_{12} \qquad 14.10$$

which gives the transformation equations

$$\begin{aligned} q_9 &= q_5 \\ q_{10} &= q_6 \\ q_{11} &= aq_5 + q_7 \\ q_{12} &= aq_6 + q_8 \\ p_5 &= p_9 + ap_{11} \\ p_6 &= p_{10} + ap_{12} \\ p_7 &= p_{11} \\ p_8 &= p_{12} \end{aligned} \qquad 14.11$$

This transformation changes the q_7 and 8 coordinates from a moving origin to a fixed origin (q_{11} and $_{12}$).

Now the Hamiltonian is

$$H = (1/2B) \left[(p_9 - q_{10} \frac{p_4}{2})^2 + (p_{10} + q_9 \frac{4}{2})^2 \right] + (1/2A)\, p_4^2$$

$$- (\mu/2)\,(q_9^2 + q_{10}^2) + (1/2m)\,(p_{11}^2 + p_{12}^2) \qquad 14.12$$

$$+ (mf^2/2) \left[(q_{11} - aq_9)^2 + (q_{12} - aq_{10})^2 \right] .$$

If we let $\lambda = q_9 + iq_{10}$ and $z = q_{11} + iq_{12}$ and note that $p_4 = A\nu = B\overline{\nu}$, the Hamilton equations give

$$\lambda'' - i\overline{\nu}\,\lambda' - (\mu/B)\lambda = (amf^2/B)\,(z - a\lambda)$$
$$z'' + f^2 z = af^2\lambda \qquad 14.13$$

Stability of this system is equivalent to requiring that the roots (r) of the determinant equation

$$\begin{vmatrix} r^2 - i\overline{\nu}r - (\mu/B) + (a^2mf^2/B) & amf^2/B \\ af^2 & r^2 + f^2 \end{vmatrix} = 0 \qquad 14.14$$

are pure imaginary numbers. This will be the case (for small m) at least if the roots for m = 0 are imaginary (usual stability) and if the roots for m = 0 are not close together (resonance). The requirement that the roots be pure imaginary is due to the form of the quantic polynomial represented by the determinant. If any root is $\alpha + i\beta$, there is also a root $-\alpha + i\beta$, and one of these roots introduces an unstable mode.

While a judicious guess could give equation 14.14, not all cases of restrained motion are as simple as the preceding. In the more complicated cases, the Hamiltonian method helps to avoid errors in the formulation of the equations of motion.

Harry L. Reed Jr.

HARRY L. REED, JR.

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